UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics — NST3PHY

MASTER OF ADVANCED STUDY IN PHYSICS — MAPY

Wednesday 17 January 2018 12.00 to 14.00

MAJOR TOPICS

Paper 1/TQM (Theories of Quantum Matter)

Answer **two** *questions only.* $\hbar = 1$ ***throughout this paper***.

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains 10 sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2×20-page answer books	Mathematical formulae handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Draft: 11th June 2018

1 In terms of boson operators satisfying $[a_{\mathbf{k}}, a^\dagger_{\mathbf{k}'}] = \delta_{\mathbf{k},\mathbf{k}'}$, Bogoliubov's Hamiltonian for the Bose gas takes the form

$$H = \sum_{\mathbf{k}\neq 0} \left[\epsilon(\mathbf{k}) a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \frac{U_0 n_0}{2} \left(a^\dagger_{\mathbf{k}} a^\dagger_{-\mathbf{k}} + a_{\mathbf{k}} a_{-\mathbf{k}} + 2 a^\dagger_{\mathbf{k}} a_{\mathbf{k}} \right) \right],$$

where $n_0 = N_0/V$ is the density of particles in the zero momentum state, $\epsilon(\mathbf{k}) = \mathbf{k}^2/2m$, and U_0 is the strength of the interaction.

(a) Show that the Hamiltonian can be diagonalized by a Bogoliubov transformation

$$b_{\mathbf{p}} = a_{\mathbf{p}} \cosh \kappa_{\mathbf{p}} + a^\dagger_{-\mathbf{p}} \sinh \kappa_{\mathbf{p}},$$

where you should find the form of $\kappa_{\mathbf{p}}$ and the Bogoliubov dispersion relation $\omega(\mathbf{p})$.

[6]

(b) Find an expression for the expectation value of the number of particles not in the condensate in terms of the mean thermal occupation of a Bogoliubov mode at temperature T

$$n_{\mathrm{B}}(\mathbf{p}) = \frac{1}{\exp(\omega(\mathbf{p})/k_{\mathrm{B}}T) - 1}.$$

Leave your answer as a sum over momenta.

[6]

(c) Show that the variance of the number of particles not in the condensate is

$$\sigma^2_{\mathbf{p}\neq 0} = \sum_{\mathbf{p}\neq 0} n_{\mathrm{B}}(\mathbf{p})(n_{\mathrm{B}}(\mathbf{p}) + 1) \left(6 \cosh^2 \kappa_{\mathbf{p}} \sinh^2 \kappa_{\mathbf{p}} + \cosh^4 \kappa_{\mathbf{p}} + \sinh^4 \kappa_{\mathbf{p}} \right) + 2 \cosh^2 \kappa_{\mathbf{p}} \sinh^2 \kappa_{\mathbf{p}}.$$

[6]

(d) By considering the low momentum behaviour of the momentum sum, show that in three dimensions $\sigma^2_{\mathbf{p}\neq 0} \to \alpha V^{4/3}$ as the volume $V \to \infty$, and give an expression for the coefficient α in terms of a momentum sum.

[6]

(e) Consider the case of zero temperature in two dimensions. How does the variance scale with system size in this case?

[6]

[*If n has probability distribution $P(n) = e^{-\beta n \epsilon}/Z$ for $n = 0, 1, 2, \ldots$, then the mean and variance are*

$$\bar{n} = \frac{1}{e^{\beta\epsilon} - 1}, \qquad \sigma^2_n = \bar{n}(\bar{n} + 1).$$
]

Solution 1

(a) This is bookwork.

$$b_{\mathbf{p}} = a_{\mathbf{p}} \cosh \kappa_{\mathbf{p}} + a^{\dagger}_{-\mathbf{p}} \sinh \kappa_{\mathbf{p}}.$$

The parameter $\kappa_{\mathbf{p}}$ of the transformation is chosen in order that there are no 'anomalous' $b^{\dagger}_{\mathbf{p}} b^{\dagger}_{-\mathbf{p}}$ or $b_{\mathbf{p}} b_{-\mathbf{p}}$ terms left in the Hamiltonian ✓✓

$$\tanh 2\kappa_{\mathbf{p}} = \frac{n_0 U_0}{\epsilon(\mathbf{p}) + n_0 U_0}. ✓✓$$

The Hamiltonian then takes the form of a sum of oscillators

$$H = E_0 + \sum_{\mathbf{p}\neq 0} \omega(\mathbf{p}) b^{\dagger}_{\mathbf{p}}. ✓ b_{\mathbf{p}}.$$

Here $\omega(\mathbf{p})$ is the Bogoliubov dispersion relation

$$\omega(\mathbf{p}) = \sqrt{\epsilon(\mathbf{p})\left(\epsilon(\mathbf{p}) + 2U_0 n_0\right)}. ✓$$

The ground state energy is not required.

(b) (Zero temperature version appears in lectures). The number of particles in state $\mathbf{p}$ is counted by the operator

$$N_{\mathbf{p}} = a^{\dagger}_{\mathbf{p}} a_{\mathbf{p}} = c^2_{\mathbf{p}} b^{\dagger}_{\mathbf{p}} b_{\mathbf{p}} + s^2_{\mathbf{p}} b_{-\mathbf{p}} b^{\dagger}_{-\mathbf{p}} - c_{\mathbf{p}} s_{\mathbf{p}} \left(b^{\dagger}_{\mathbf{p}} b^{\dagger}_{-\mathbf{p}} + b_{-\mathbf{p}} b_{\mathbf{p}} \right). ✓✓$$

The expectation value in an eigenstate of H, with definite occupations of the Bogoliubov modes, is

$$\langle N_{\mathbf{p}} \rangle = c^2_{\mathbf{p}} n_{\mathrm{B}}(\mathbf{p}) + s^2_{\mathbf{p}} (n_{\mathrm{B}}(\mathbf{p}) + 1). ✓✓$$

Where $c_{\mathbf{p}} = \cosh \kappa_{\mathbf{p}}$, $s_{\mathbf{p}} = \sinh \kappa_{\mathbf{p}}$. Taking the thermal average, the mean number of particles not in the condensate is

$$N_{\mathrm{nc}} = \sum_{\mathbf{p}\neq 0} \left[c^2_{\mathbf{p}} \bar{n}_{\mathrm{B}}(\mathbf{p}) + s^2_{\mathbf{p}} (\bar{n}_{\mathrm{B}}(\mathbf{p}) + 1) \right]. ✓✓$$

(c) Using the above expression for $N_{\mathbf{p}}$, we find

$$\langle N_{\mathbf{p}} N_{\mathbf{p}'} \rangle = \left[c^2_{\mathbf{p}} n_{\mathrm{B}}(\mathbf{p}) + s^2_{\mathbf{p}} (n_{\mathrm{B}}(\mathbf{p}) + 1) \right] \left[c^2_{\mathbf{p}'} n_{\mathrm{B}}(\mathbf{p}') + s^2_{\mathbf{p}'} (n_{\mathrm{B}}(\mathbf{p}') + 1) \right] ✓ \tag{1}$$

$$+ s_{\mathbf{p}} c_{\mathbf{p}} s_{\mathbf{p}'} c_{\mathbf{p}'} \langle b^{\dagger}_{\mathbf{p}} b^{\dagger}_{-\mathbf{p}} b_{-\mathbf{p}'} b_{\mathbf{p}'} + b_{-\mathbf{p}} b_{\mathbf{p}} b^{\dagger}_{\mathbf{p}'} b^{\dagger}_{-\mathbf{p}'} \rangle. ✓ \tag{2}$$

We also have

$$\langle b^{\dagger}_{\mathbf{p}} b^{\dagger}_{-\mathbf{p}} b_{-\mathbf{p}'} b_{\mathbf{p}'} \rangle = n_{\mathrm{B}}(\mathbf{p}) n_{\mathrm{B}}(-\mathbf{p}), \qquad \mathbf{p} = \pm \mathbf{p}'. ✓$$

and

$$\langle b_{-\mathbf{p}'} b_{\mathbf{p}'} b^\dagger_{\mathbf{p}} b^\dagger_{-\mathbf{p}} \rangle = [n_{\mathrm{B}}(\mathbf{p}) + 1][n_{\mathrm{B}}(-\mathbf{p}) + 1], \qquad \mathbf{p} = \pm\mathbf{p}'.\checkmark$$

In this way, we express everything in terms of the occupancy. Finally, we perform the thermal average, taking the hint that $\overline{n^2_{\mathrm{B}}}(\mathbf{p}) - \bar{n}_{\mathrm{B}}(\mathbf{p})^2 = \bar{n}_{\mathrm{B}}(\mathbf{p})(\bar{n}_{\mathrm{B}}(\mathbf{p}) + 1)$

$$\operatorname{Var} N_{\mathrm{nc}} = \sum_{\mathbf{p},\mathbf{p}'\neq 0} \overline{\langle N_{\mathbf{p}} N_{\mathbf{p}'} \rangle} - \left(\sum_{\mathbf{p}\neq 0} \overline{\langle N_{\mathbf{p}} \rangle} \right)^2 \checkmark \tag{3}$$

$$= \sum_{\mathbf{p}\neq 0} \left[\left(c^2_{\mathbf{p}} + s^2_{\mathbf{p}} \right)^2 + 4 s^2_{\mathbf{p}} c^2_{\mathbf{p}} \right] \bar{n}_{\mathrm{B}}(\mathbf{p})(\bar{n}_{\mathrm{B}}(\mathbf{p}) + 1) + 2 s^2_{\mathbf{p}} c^2_{\mathbf{p}}.\checkmark \tag{4}$$

(d) At low momenta

$$e^{\kappa_{\mathbf{p}}} \to \left(\frac{n_0 U_0}{\epsilon(\mathbf{p})} \right)^{1/4} \checkmark, \qquad n_{\mathrm{B}}(\mathbf{p}) \to \frac{1}{\beta\omega(\mathbf{p})} = \frac{1}{\beta c |\mathbf{p}|}.\checkmark \tag{5}$$

The low momentum contribution to the variance then takes the simple form

$$\operatorname{Var} N_{\mathrm{nc}} = \frac{8 n_0 U_0}{\beta^2 c^2} \sum_{\mathbf{p}\neq 0} \frac{1}{\epsilon(\mathbf{p})^2 |\mathbf{p}|^2} \checkmark \tag{6}$$

$$= \frac{16 m^2}{\beta^2} \sum_{\mathbf{p}\neq 0} \frac{1}{|\mathbf{p}|^4} \checkmark \tag{7}$$

$$= \frac{m^2 L^4}{\pi^4 \beta^2} \sum_{p,q,r\neq 0} \frac{1}{|p^2 + q^2 + r^2|^2} \checkmark. \tag{8}$$

Since $L^4 = V^{4/3}$, the coefficient α is

$$\alpha = \frac{m^2}{\pi^4 \beta^2} \sum_{p,q,r\neq 0} \frac{1}{|p^2 + q^2 + r^2|^2}.\checkmark \tag{9}$$

(e) Only the last term in the variance contributes✓✓. In the low momentum limit we get

$$\operatorname{Var} N_{\mathrm{nc}} = \sum_{\mathbf{p}\neq 0} 2 s^2_{\mathbf{p}} c^2_{\mathbf{p}} \checkmark \tag{10}$$

$$= 2 \sum_{\mathbf{p}\neq 0} \frac{n_0 U_0}{\epsilon(\mathbf{p})} \checkmark. \tag{11}$$

This is logarithimically divergent in 2D✓, so $\operatorname{Var} N_{\mathrm{nc}} \propto L^2 \log L$.✓

2 The fundamental commutator of a boson field in one dimension is

$$[\psi(x), \psi^\dagger(y)] = \delta(x-y).$$

(a) Show that this relation is reproduced if

$$\psi(x) = e^{i\theta(x)}\sqrt{n(x)},$$

and $n(x)$ and $\theta(x)$ satisfy

$$[n(x), \theta(y)] = i\delta(x-y).$$

[7]

(b) An approximate quadratic Hamiltonian describing a gas of interacting bosons of density n_0 is

$$H = \frac{1}{2}\int dx \left[\frac{n_0}{m}(\partial_x\theta)^2 + U_0(\rho - n_0)^2\right].$$

Find the spectrum by writing the Hamiltonian in terms of the Fourier modes

$$\rho(x) = \frac{1}{\sqrt{L}}\sum_p \rho_p e^{ipx},$$
$$\theta(x) = \frac{1}{\sqrt{L}}\sum_p \theta_p e^{ipx},$$

with $p = 2\pi n/L$ for n integer, $\rho_p^\dagger = \rho_{-p}$, and $\theta_p^\dagger = \theta_{-p}$. [6]

(c) Show that the ground state wavefunction may be written

$$|\Psi\rangle = \exp\left[-\sum_{q\neq 0}\frac{mc}{2n_0|q|}\rho_q\rho_{-q}\right]|\theta_p = 0,\ \text{all } p\rangle,$$

where you should give an expression for the speed of sound c. [7]

(d) Convert the result into a wavefunction for N particles by using the expression for ρ_q in terms of the particle coordinates

$$\rho_q = \sum_{j=1}^{N} e^{-iqx_j}.$$

[$|\theta_p = 0,\ \textit{all } p\rangle$ *corresponds to a constant wavefunction.*]

[4]

(e) For what value of the parameter $K = \pi n_0/mc$ is this system equivalent to a system of free fermions? Why? [6]

Solution 2

(a) This is bookwork. It's easier if we discretize space so that

$$\left[n_j, \theta_k\right] = i\delta_{jk}.\checkmark \tag{12}$$

We see that the operators certainly commute at different sites and at the same site we have

$$\begin{aligned}\left[\psi_j, \psi_j^\dagger\right] &= e^{i\theta_j} n_j e^{-i\theta_j} - n_j \checkmark\checkmark &(13)\\ &= n_j + 1 - n_j = 1.\checkmark\checkmark &(14)\end{aligned}$$

We see that $e^{i\theta_j}$ acts to increment n_j by one unit. In the continuum limit we recover the required answer.✓✓

(b) Also bookwork. Substitution into the Hamiltonian yields

$$H = \frac{1}{2}\sum_{p\neq 0} \frac{n_0 p^2}{m}\theta_{-p}\theta_p + U_0\rho_{-p}\rho_p.\checkmark\checkmark \tag{15}$$

(Note that $n_0 = \rho_0$). Comparison with the Hamiltonian of an oscillator then yields✓✓

$$H = \sum_p \frac{\omega_p}{2}\left[a_p^\dagger a_p + a_p a_p^\dagger\right], \tag{16}$$

with $\omega_p = (n_0 U_0/m)^{1/2}|p|$✓✓.

(c) Each factor is the Gaussian ground state wavefunction✓✓. Observe that it is annihilated by

$$a_p = \frac{1}{\sqrt{2}}\left[\sqrt{\frac{mc}{n_0|p|}}\rho_p + i\sqrt{\frac{n_0|p|}{mc}}\theta_{-p}\right].\checkmark\checkmark\checkmark \tag{17}$$

If we identify $c^2 = n_0 U_0/m$✓. Note that the sum in the Gaussian gives rise to identical factors for $\pm p$).✓

(d) By using the given form for the density✓✓. we obtain the wavefunction

$$\Psi(x_1, \ldots N) = \exp\left[-\sum_{q,j,k} \frac{mc}{2n_0|q|} e^{iq(x_j - x_k)}\right].\checkmark\checkmark \tag{18}$$

Discussion of the behaviour of this function can be deferred to the next part.

(e) Some discussion of the following asymptote (discussed in the lectures in a different but related context)

$$\sum_{q\neq 0} \frac{mc}{2n_0|q|} e^{iqx} \to -\frac{mc}{2\pi n_0}\log|x|/L, \qquad |x| \ll L\checkmark\checkmark \tag{19}$$

This means that

$$\Psi(x_1, \ldots N) \sim \prod_{j<k} |x_j - x_k|^{K^{-1}}. \checkmark\checkmark \tag{20}$$

$K = 1$ then corresponds to (modulus of) the ground state wavefunction of free fermions.✓✓

A less satisfactory answer would be to evaluate $K = \pi n_0/mc$ with $c = v_{\mathrm{F}}$, which also gives $K = 1$.

3 Consider the tight-binding model describing fermions without spin moving in one dimension

$$H_t = -t \sum_{j=1}^{N} \left[a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1} \right].$$

a_j, $a_j^\dagger$ satisfy the usual anticommutation relations. Assume periodic boundary conditions, so that $a_{j+N} = a_j$.

(a) Show that the single particle states of the model are plane waves and find their dispersion. [6]

(b) Now consider

$$H_{t_1 t_2} = -t_2 \sum_{j=1}^{N/2} \left[a_{2j+1}^\dagger a_{2j} + a_{2j}^\dagger a_{2j+1} \right] - t_1 \sum_{j=1}^{N/2} \left[a_{2j}^\dagger a_{2j-1} + a_{2j-1}^\dagger a_{2j} \right].$$

Show that in this case the single particle states consist of two bands with dispersion

$$\omega_\pm(\eta) = \pm \sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos 2\eta}.$$

Sketch the bands for $|t_1 - t_2| << t_{1,2}$. [7]

(c) Find the ground state energy of a system of $N/2$ fermions with Hamiltonian $H_{t_1 t_2}$ in the limit $N \to \infty$, leaving your answer as an integral. [4]

(d) The sites in the lattice are located at $x_j = ja + u_j$. Displacements $u_j = (-1)^j u$ give rise to variations $t_{1,2} = t \mp t' u$ and an elastic energy $\frac{Nk}{2} u^2$, for constants t' and k.

By combining the ground state energy of the fermions from part (c) with the elastic energy, find an equation for u that minimizes the total energy. Explain why this equation always has a solution for nonzero u. [7]

(e) Discuss the effect of finite temperature on your conclusion to (d). [6]

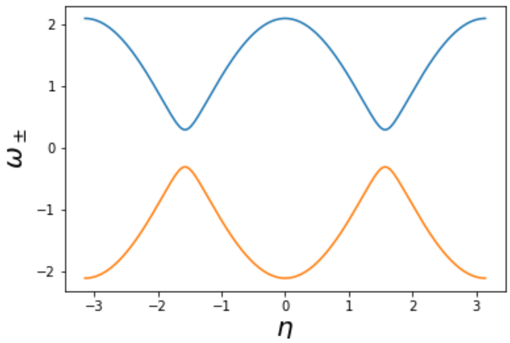

Figure 1: The bands $\omega_{\pm}(\eta)$

Solution 3

(a) This is bookwork. Substitute

$$a_j = \sum_n \alpha_n e^{i\eta_n j}, \checkmark\checkmark \tag{21}$$

with $\eta_n = 2\pi n/N$. After performing the $\sum_{j=1}^{N}$ ✓the result is

$$H_t = \sum_n \omega(\eta)\alpha_n^\dagger \alpha_n, \checkmark \tag{22}$$

with $\omega(\eta) = -2t\cos(\eta)$.✓✓

(b) The important thing here is to realize that we now have two sites in the unit cell ✓, so we write

$$a_{2j-1} = \sum_n \alpha_n e^{i\eta_n j}, \tag{23}$$

$$a_{2j} = \sum_n \beta_n e^{i\eta_n j}, \checkmark \tag{24}$$

where $\eta = 4\pi n/N$ for $n = 0, \pm 1 \ldots \pm (N/2-1)/2$ (if $N/2$ odd)✓. The result is a Hamiltonian of the form

$$H_{t_1 t_2} = -\sum_n \begin{pmatrix} \alpha_n^\dagger & \beta_n^\dagger \end{pmatrix} \begin{pmatrix} 0 & t_1 + t_2 e^{2i\eta_n} \\ t_1 + t_2 e^{-2i\eta_n} & 0 \end{pmatrix} \begin{pmatrix} \alpha_n \\ \beta_n \end{pmatrix} . \checkmark\checkmark \tag{25}$$

This can be diagonalized by taking symmetric and antisymmetric components, yielding the given bands.✓. Sketch ✓

(c) This is just a question of filling the lower band ✓✓

$$E_0(t_1, t_2) = \sum_n \omega_-(\eta_n)\checkmark \to -\frac{N}{2}\int_{-\pi}^{\pi} \frac{d\eta}{2\pi}\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos(2\eta)}.\checkmark \tag{26}$$

(d) We have

$$\omega_{-}(\eta) = -2\sqrt{t^2\cos^2\eta + (t'u)^2\sin^2\eta}, \tag{27}$$

Finding the extremum of $\frac{Nku^2}{2} + E_0(t_1, t_2)$✓ w.r.t u gives the

$$k = \int_{-\pi}^{\pi} \frac{d\eta}{2\pi} \frac{t'^2}{\sqrt{t^2\cos^2\eta + (t'u)^2\sin^2\eta}}.✓ \tag{28}$$

The integral doesn't look straightforward, but we can proceed by seeing that for small u there is a log divergence at $\eta = \pm\pi/2$ that is cut off by finite u✓✓. In the vicinity of these points the integrand becomes

$$\frac{1}{2\pi} \frac{t'^2}{\sqrt{t^2(\eta \mp \pi/2)^2 + (t'u)^2}}.✓ \tag{29}$$

On account of the $\log|u|$ dependence of the integral (c.f. BCS self-consistent equation), one can find a solution with finite u no matter the magnitude of k.✓✓

(e) Thermal depopulation of the lower band ✓(and occupation of the upper band✓) gives rise to factor of $2n_{\mathrm{F}}[\omega_(\eta)] - 1$, where $n_{\mathrm{F}}(\omega)$ is the Fermi–Dirac distribution✓. This leads to the equation

$$k = \int_{-\pi}^{\pi} \frac{d\eta}{2\pi} \frac{t'^2\tanh[\omega_{+}(\eta)/2k_{\mathrm{B}}T]}{\sqrt{t^2\cos^2\eta + (t'u)^2\sin^2\eta}}.✓✓ \tag{30}$$

This cuts off the logarithmic divergence of the integral, leading to a finite transition temperature.✓

END OF PAPER