UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics — NST3PHY
MASTER OF ADVANCED STUDY IN PHYSICS — MAPY

Wednesday 16th January 2019 12.00 to 14.00

MAJOR TOPICS
Paper 1/TQM (Theories of Quantum Matter)

Answer **two** *questions only.* $\hbar = 1$ ***throughout this paper***.

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains 11 sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
2 × 20-page answer books	Mathematical formulae handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

Draft: 9th November 2020

1

(a) Show that for operators A and B

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \ldots$$

[6]

The antiferromagnetic Heisenberg chain of N spins has Hamiltonian

$$H = J \sum_{j=1}^{N} \mathbf{s}_j \cdot \mathbf{s}_{j+1},$$

with $J > 0$, periodic boundary conditions ($\mathbf{s}_j = \mathbf{s}_{j+N}$), and where the components of $\mathbf{s}_j = (s^x_j, s^y_j, s^z_j)$ of $\mathbf{s}_j$ satisfy

$$\left[s^a_j, s^b_k\right] = i\epsilon_{abc} s^c \delta_{jk}.$$

(b) Find an explicit expression for $H' = UHU^\dagger$, where the unitary transformation U is

$$U = \exp\left(\frac{2\pi i}{N} \sum_{j=1}^{N} j s^z_j\right).$$

[10]

(c) It can be shown that a translationally invariant ground state $|0\rangle$ of H obeys

$$\langle 0|U|0\rangle = \langle 0|\exp\left(\frac{2\pi i}{N} \sum_{j=1}^{N} j s^z_{j-1}\right)|0\rangle.$$

If $|0\rangle$ is an eigenstate of $S^z = \sum_{j=1}^{N} s^z_j$ with eigenvalue zero and the spins have half-integer spin (i.e. $s = 1/2, 3/2$, etc.), explain why it follows that $\langle 0|U|0\rangle = 0$.

[7]

(d) Treating $U^\dagger|0\rangle$ as a variational state for the half-integer case, explain how the results of parts (b) and (c) may be combined to find an upper bound on the gap between the ground and first excited states. How does this upper bound depend on N as $N \to \infty$?

[7]

Solution 1

This question is based on Affleck, I. & Lieb, E.H. Lett. Math. Phys. (1986) 12: 57.

(a) This is the *Hadamard lemma*. If $B(x) = e^{xA}Be^{-xA}$ then

$$B'(x) = [A, B(x)]\,. \tag{1}$$

Integrating gives

$$B(x) = B + \int_0^x [A, B(x)] \tag{2}$$

Since $B(0) = B$. Iterating this equation produces

$$B(x) = B + x\,[A, B] + \frac{x^2}{2!}\,[A, [A, B]] + \frac{x^3}{3!}\,[A, [A, [A, B]]] + \ldots, \tag{3}$$

from which the result follows for $x = 1$.

(b) First observe that

$$H' = J\sum_{j=1}^{N} \mathbf{s}'_j \cdot \mathbf{s}'_{j+1}, \tag{4}$$

where $\tilde{\mathbf{s}}_j = U\mathbf{s}_jU^\dagger$ is the transformed spin operator. By the Hadamard Lemma, only the x and y components are altered. Explicit calculation gives

$$\tilde{s}^x_j = \cos\left(\frac{2\pi j}{N}\right)s^x_j - \sin\left(\frac{2\pi j}{N}\right)s^y_j \tag{5}$$

$$\tilde{s}^y_j = \cos\left(\frac{2\pi j}{N}\right)s^y_j + \sin\left(\frac{2\pi j}{N}\right)s^x_j. \tag{6}$$

or more compactly,

$$\tilde{s}^+_j = e^{2\pi i j/N}s^+_j \tag{7}$$

The transformed Hamiltonian is then

$$H' = J\sum_{j=1}^{N}\left[\frac{1}{2}\left(e^{-2\pi i/N}s^+_j s^-_{j+1} + e^{2\pi i/N}s^-_j s^+_{j+1}\right) + s^z_j s^z_{j+1}\right]. \tag{8}$$

(c) Comparing the two expressions for $\langle 0|U|0\rangle$

$$\langle 0|\exp\left(\frac{2\pi i}{N}\sum_{j=1}^{N} js^z_j\right)|0\rangle = \langle 0|\exp\left(\frac{2\pi i}{N}\sum_{j=1}^{N} js^z_{j-1}\right)|0\rangle. \tag{9}$$

We find that

$$\langle 0|U|0\rangle = \langle 0|Ue^{2\pi i S^z/N}e^{2\pi i s^z_N}|0\rangle = (-1)^{2s}\langle 0|U|0\rangle \tag{10}$$

where we have used the fact that $|0\rangle$ is an eigenstate of $S^z = \sum_j s^z_j$ with eigenvalue zero, and that rotating a spin-s state by 2π yields a factor $(-1)^{2s}$. For half integer spin, we must have $\langle 0|U|0\rangle = 0$.

(d) The argument is

- The state $U^\dagger|0\rangle$ is orthogonal to $|0\rangle$.
- By the variational principle, the expectation of the energy in this state is an upper bound on the energy of the first excited state.
- The expectation value is

$$\langle 0|UHU^\dagger|0\rangle = \langle 0|H'|0\rangle. \tag{11}$$

- Therefore the energy gap is upper bounded by

$$\langle 0|H'|0\rangle - \langle 0|H|0\rangle = J\langle 0|\sum_{j=1}^{N}\frac{1}{2}\left([e^{-2\pi i/N}-1]s_j^+ s_{j+1}^- + [e^{2\pi i/N}-1]s_j^- s_{j+1}^+\right)|0\rangle \tag{12}$$

$$= \frac{J}{2}\left(\cos\left(\frac{2\pi}{N}\right)-1\right)\langle 0|\sum_{j=1}^{N}\left(s_j^+ s_{j+1}^- + s_j^- s_{j+1}^+\right)|0\rangle \tag{13}$$

$$\propto \frac{1}{N}, \text{ as } N\to\infty. \tag{14}$$

In the first step we use $\sum_j \langle s_j^+ s_{j+1}^-\rangle = \sum_j \langle s_j^- s_{j+1}^+\rangle$, which follows from symmetry under $j \to -j$ (The paper gives a slightly different argument). Then we use translational invariance to say that the expectation value is $\propto N$.

2 An elastic chain consisting of N coupled masses is described by the Hamiltonian

$$H = \sum_{j=1}^{N} \left[\frac{p_j^2}{2m} + \frac{k}{2}(u_j - u_{j+1})^2 \right].$$

where $\left[u_j, p_k\right] = i\delta_{jk}$ are canonically conjugate variables and periodic boundary conditions $u_j = u_{j+N}$, $p_j = p_{j+N}$ apply.

(a) By solving the *classical* equations with an external force acting on mass $j = 0$, show that in the limit $N \to \infty$ the susceptibility $\chi(\omega)$ of the coordinate u_0 has the form

$$\chi(\omega) = \frac{1}{m} \int_{-\pi}^{\pi} \frac{d\eta}{2\pi} \frac{1}{\Omega^2(\eta) - (\omega + i0)^2},$$

where you should find an expression for the dispersion relation $\Omega(\eta)$.

[9]

(b) Find an explicit formula for the imaginary part of the susceptibility.

[*Recall that:*

$$Im \frac{1}{x \mp i0} = \pm\pi\delta(x),$$

]

[6]

(c) Ignoring the motion of the centre of mass, the displacement u_0 of the zeroth mass can be written in terms of creation ($a_n^\dagger$) and annihilation (a_n) operators of phonons of wavevector $\eta_n = 2\pi n/N$ as

$$u_j(t) = \frac{1}{\sqrt{N}} \sum_{\substack{n \neq 0 \\ |n| \leq (N-1)/2}} q_n(t) e^{i\eta_n j},$$

$$q_n = \sqrt{\frac{1}{2m\omega(\eta_n)}} \left(a_n + a_{-n}^\dagger\right).$$

Evaluate the quantum noise spectrum (at zero temperature)

$$S(\omega) = \int_{-\infty}^{\infty} \langle 0|u_0(t)u_0(0)|0\rangle \, e^{i\omega t} \, dt.$$

where $|0\rangle$ is the ground state of the chain. Verify the zero temperature fluctuation dissipation relation

$$S(\omega) = 2\mathrm{Im}\chi(\omega).$$

[7]

(d) Find the behaviour of $\langle 0|u_0^2|0\rangle$ in the $N \to \infty$ limit, and compare with the fluctuations of the variable $X \equiv u_0 - (u_1 + u_{-1})/2$.

[8]

Solution 2

This question is based on Exercise 1, Chapter 11, of *Advanced Quantum Mechanics* by Nazarov and Danon (CUP, 2013).

(a) The classical equations of motions are

$$m\ddot{u}_j + k\left(2u_j - u_{j-1} - u_{j+1}\right) = f(t)\delta_{j,0}. \tag{15}$$

For a harmonic force $f(t) = f_\omega e^{-i\omega t}$ we get

$$-m\omega^2 u_j + k\left(2u_j - u_{j-1} - u_{j+1}\right) = f_\omega \delta_{j,0}. \tag{16}$$

Expanding the displacement in Fourier modes

$$u_j = \sum_\eta e^{i\eta j} q_\eta \tag{17}$$

gives

$$\frac{f_\omega}{N} = \left[-m\omega^2 + 2k(1-\cos\eta)\right] q_\eta(\omega), \tag{18}$$

and a susceptibility of

$$\chi(\omega) = \frac{1}{mN}\sum_\eta \frac{1}{\Omega^2(\eta) - (\omega + i0)^2} \tag{19}$$

where the dispersion $\Omega(\eta) = \Omega|\sin(\eta/2)|$, where $\Omega_0 \equiv \frac{4k}{m}$ is the maximum phonon frequency. Taking the $N \to \infty$ limit

$$\sum_\eta (\cdots) = N\int_{-\pi}^{p} i\frac{d\eta}{2\pi}(\cdots) \tag{20}$$

gives the stated answer.

(b) Applying the given formula to give a delta-function that allows the integral to be done, giving

$$\operatorname{Im}\chi(\omega) = \frac{1}{m\Omega_0\omega}\frac{1}{\sqrt{1-(\omega/\Omega_0)^2}} \tag{21}$$

(c) Use the time dependence $a_n(t) = a_n e^{-i\omega(\eta_n)t}$, the given relations, and the fact that the only surviving contribution is

$$\langle 0|a_n(t)a_n^\dagger(0)|0\rangle = e^{-i\omega(\eta_n)t}. \tag{22}$$

This gives $S(\omega)$ as the integral

$$S(\omega) = \frac{1}{N}\sum_{\eta\neq 0}\int_{-\infty}^{\infty}\frac{e^{i(\omega+i0-\Omega(\eta))t}}{2m\Omega(\eta)}dt = \frac{1}{N}\sum_{\eta\neq 0}\frac{2\pi\delta(\omega-\Omega(\eta))}{2m\Omega(\eta)} \longrightarrow \int_{-\pi}^{\pi}\frac{\delta(\omega-\Omega(\eta))}{2m\Omega(\eta)}d\eta \tag{23}$$

$$= \frac{2}{m\Omega_0\omega}\frac{1}{\sqrt{1-(\omega/\Omega_0)^2}} \tag{24}$$

which verifies the FDT.

(d) A very similar calculation (or just the ω integral of the previous calculation) gives

$$\langle 0|u_0^2|0\rangle = \frac{1}{N}\sum_{\eta\neq 0}\frac{1}{2m\Omega(\eta)}. \tag{25}$$

This is logarithmically divergent in the $N \to \infty$ limit. Specifically

$$\langle 0|u_0^2|0\rangle \sim \frac{1}{\pi m\Omega_0}\log N \tag{26}$$

The same calculation for the variable $X \equiv u_0 - (u_1 + u_{-1})/2$ gives

$$\langle 0|u_0^2|0\rangle = \frac{1}{N}\sum_{\eta}\frac{1-\cos\eta}{2m\Omega(\eta)}. \tag{27}$$

On account of the numerator vanishing as $\eta \to 0$, this isn't divergent. That's all I'm looking for.

3 The Hamiltonian for fermions hopping on a chain of N sites in the presence of a pairing term is

$$H = \sum_{j=1}^{N} \left(-t \left[a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j \right] - \mu \left[a_j^\dagger a_j - \frac{1}{2} \right] + \Delta a_j a_{j+1} + \Delta^* a_{j+1}^\dagger a_j^\dagger \right),$$

where $\Delta = |\Delta| e^{i\theta}$. a_j and $a_j^\dagger$ satisfy the usual anticommutation relations

$$\left\{ a_j, a_k^\dagger \right\} = \delta_{jk}.$$

Do not assume periodic boundary conditions.

(a) $2N$ operators are defined as

$$\begin{aligned} c_{2j-1} &= e^{i\theta/2} a_j + e^{-i\theta/2} a_j^\dagger \\ c_{2j} &= -ie^{i\theta/2} a_j + ie^{-i\theta/2} a_j^\dagger. \end{aligned}$$

Show that these operators satisfy the anticommutation relations

$$\left\{ c_j, c_k \right\} = 2\delta_{jk},$$

and express the Hamiltonian in terms of these operators.

[6]

(b) Describe the ground state of the model for $t = |\Delta| = 0$ for $\mu < 0$ in terms of the original $a_j^\dagger$, a_j variables.

[5]

(c) Describe the ground state(s) of the model for $t = |\Delta| > 0$ for $\mu = 0$ in terms of the new variables

$$\tilde{a}_j = \frac{1}{2} \left(c_{2j} + ic_{2j+1} \right) \qquad j = 1, \ldots N-1.$$

Explain why there are two ground states and how they differ.

[8]

(d) New operators b_j are defined in terms of a $2N \times 2N$ matrix W as

$$\begin{pmatrix} b_1 \\ b_2 \\ \cdots \\ b_{2N} \end{pmatrix} = W \begin{pmatrix} c_1 \\ c_2 \\ \cdots \\ c_{2N} \end{pmatrix}.$$

What condition must W satisfy for the b_j to have the anticommutation relations

$$\left\{ b_j, b_k \right\} = 2\delta_{jk}?$$

[4]

(e) Show that for $t = |\Delta|$ the operators

$$b_1 = \sum_{j=1}^{N} x^j c_{2j-1}$$
$$b_2 = \sum_{j=1}^{N} x^{-j} c_{2j}$$

commute with the Hamiltonian for a particular value of x, which you should find. What can you conclude about these modes for $2t > |\mu|$ and $N \to \infty$?

[7]

Solution 3

This question is based on A Yu Kitaev, Usp. Fiz. Nauk (Suppl.) 171 (10) (2001).

(a) The first part follows from the canonical anticommutation relations

$$\left\{a_j, a_k^\dagger\right\} = \delta_{jk}. \tag{28}$$

Next, inverting the definition gives

$$a_j = \frac{e^{-i\theta/2}}{2}\left(c_{2j-1} + ic_{2j}\right).$$

Substituting into the Hamiltonian yields

$$H = \frac{i}{2}\sum_j \left(-\mu c_{2j-1}c_{2j} + [t + |\varDelta|]\, c_{2j}c_{2j+1}\, [-t + |\varDelta|]\, c_{2j-1}c_{2j+2}\right) \tag{29}$$

(b) The key is to note that in terms of the original variables

$$H = -\mu \sum_{j=1}^{N}\left[a_j^\dagger a_j - \frac{1}{2}\right]. \tag{30}$$

Thus for $\mu < 0$ the ground state corresponds to an empty vacuum with no fermions i.e. $a_j^\dagger a_j = 0$ for all j.

(c) In this case the Hamiltonian has the form

$$H = it\sum_{j=1}^{N-1} c_{2j}c_{2j+1}. \tag{31}$$

In terms of the tilded variables this is

$$H = 2t\sum_{j=1}^{N-1}\left[\tilde{a}^\dagger{}_j\tilde{a}_j - \frac{1}{2}\right]. \tag{32}$$

Thus, for $t > 0$, the ground state is a vacuum of the $N-1$ fermions $\tilde{a}_j$ $j = 1, \ldots N-1$. Since c_1 and c_{2N} do not appear in the Hamiltonian, two ground states can be distinguished based on the presence or absence of a fermion $(c_1 + ic_{2L})/2$.

(d) By direct substitution one finds the condition $W^T W = 1$ i.e. W must be orthogonal.

(e) Calculating the commutator gives

$$[H, b_1] = \frac{i}{2}\sum_j \left(-\mu x^j + [t + |\varDelta|]\, x^{j+1}\right) c_{2j}, \tag{33}$$

so that for $x = -\mu/(t + |\varDelta|) = -\mu/(2t)$ the right hand side vanishes. Similarly for b_2. Conclusion: for $|\mu| < 2t$ we have $|x| < 1$ and the two modes are localized near the two ends of the system (strictly applies to $N \to \infty$ only as one otherwise has to impose a boundary condition at the other end).

END OF PAPER