UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics — NST3PHY
MASTER OF ADVANCED STUDY IN PHYSICS — MAPY

Thursday 21st January 2021, 10:00-12:00

MAJOR TOPICS
Paper 1/TQM (Theories of Quantum Matter)

Answer **two** *questions only.* $\hbar = 1$ ***throughout this paper***.

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains 4 sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.

You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
linear graph paper	Mathematical Formulae Handbook
Rough workpad	Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V8.4

A

1 (a) A system is described by a Hamiltonian that switches between two values

$$H_\pm = \overbrace{\frac{p^2}{2m} + \frac{1}{2}kx^2}^{\equiv H_{\rm SHO}} \pm k\xi x,$$

where $[x, p] = i$ and ξ is a parameter with units of length. Show that the eigenstates of the Hamiltonian $H_\pm$ are

$$|n, \pm\rangle = \exp(\pm i p\xi)|n\rangle,$$

where $|n\rangle$ are the eigenstates of $H_{\rm SHO}$. Find the corresponding eigenvalues. [8]

(b) The probability to observe the system in state $|n, -\rangle$, if it starts in $|0, +\rangle$, is $p_n = |\langle n, -|0, +\rangle|^2$. Show that these probabilities are described by a Poisson distribution $p_n = \frac{e^{-\lambda}\lambda^n}{n!}$, identifying the parameter λ. [8]

[*You may find the following formula helpful. If* $[a, a^\dagger] = 1$,

$$\exp(\eta(a - a^\dagger)) = \exp(-\eta a^\dagger)\exp(\eta a)\exp(-\eta^2/2).$$

For the harmonic oscillator $a = \sqrt{\frac{m\omega}{2}}x + i\sqrt{\frac{1}{2m\omega}}p$]

(c) An elastic chain of N (odd) masses is described by the Hamiltonian

$$H_\pm = \sum_{j=-(N-1)/2}^{(N-1)/2}\left[\frac{p_j^2}{2m} + \frac{1}{2}k(x_{j+1} - x_j)^2\right] \pm k\xi(x_1 - x_{-1}),$$

with periodic boundary condition $x_{j+N} = x_j$. The chain can be described by the normal mode expansion

$$x_j = \frac{q_0'}{\sqrt{N}} + \sqrt{\frac{2}{N}}\sum_{n=1}^{(N-1)/2}\left[q_n'\cos(\eta_n j) + q_n''\sin(\eta_n j)\right], \qquad \eta_n = \frac{2\pi n}{N},$$

with a similar expression for p_j. Show that the last term shifts q_n'' by

$$\xi_n = \pm\xi\sqrt{\frac{2}{N}}\frac{\sin(\eta_n)}{1 - \cos(\eta_n)}.$$

[7]

(d) Initially, the Hamiltonian is H_+ and the system starts in the ground state. The Hamiltonian then abruptly changes to H_-. Find an expression for the probability that the system is found in the ground state of H_- (i.e. without creating any phonons). Analyze the behaviour of this expression as $N \to \infty$. [7]

Solution 1

(a) This is a matter of completing the square

$$H = \frac{p^2}{2m} + \frac{1}{2}k(x+\xi\sigma_z)^2 - \frac{1}{2}k\xi^2.$$

The eigenstates are shifted to the left and right by an amount ξ, which is accomplished by $e^{\pm ip\xi}$. The eigenvalues are $(n+\frac{1}{2})\omega - \frac{1}{2}k\xi^2$.

Although I'd be happy with the statement that $e^{\pm ip\xi}$ is a translation operator and this does the job, here's an explicit check that the given states are eigenvalues. Use the representation $x = i\frac{d}{dp}$ to give

$$xe^{ip\xi} = -\xi e^{ip\xi} + e^{ip\xi}x \tag{1}$$
$$x^2 e^{ip\xi} = \xi^2 e^{ip\xi} - 2\xi e^{ip\xi}x - e^{ip\xi}x^2, \tag{2}$$

which gives us

$$\left(\frac{p^2}{2m} + \frac{1}{2}kx^2 + k\xi x\right)e^{ip\xi} = e^{ip\xi}\left(\frac{p^2}{2m} + \frac{1}{2}kx^2 - \frac{1}{2}k\xi^2\right)$$

(b) We have $p_n = |\langle -, n|+, 0\rangle|^2 = |\langle n|e^{2ip\xi}|0\rangle|^2$ and

$$\langle n|e^{2ip\xi}|0\rangle = \frac{1}{\sqrt{n!}}\langle 0|a^n e^{\sqrt{2m\omega}\xi(a-a^\dagger)}|0\rangle \tag{3}$$
$$= \frac{e^{-\xi^2 m\omega}}{\sqrt{n!}}\langle 0|a^n e^{-\xi\sqrt{2m\omega}a^\dagger}e^{\xi\sqrt{2m\omega}a}|0\rangle \tag{4}$$
$$= \frac{e^{-\xi^2 m\omega}}{\sqrt{n!}}\left(\xi\sqrt{2m\omega}\right)^n \tag{5}$$

where we used the stated results and the coherent state property $ae^{\alpha a^\dagger}|0\rangle = \alpha e^{\alpha a^\dagger}|0\rangle$. p_n is then Poisson with rate $\lambda = 2\xi^2 m\omega$.

(c) By comparing the coupling with the potential energy

$$\sum_n k(1-\cos(\eta_n))\left[q_n' q_n' + q_n'' q_n''\right] + 2k\xi\sqrt{\frac{2}{N}}\sin(\eta_n)q_n''$$

The variable q_n'' gets shifted by an amount

$$\xi_n = \pm\xi\sqrt{\frac{2}{N}}\frac{\sin(\eta_n)}{1-\cos(\eta_n)}$$

.

(d) Using the previous result the distribution of quanta in each mode after transition is Poissonian with rates

$$\lambda_n = \frac{4m\omega(\eta)\xi^2}{N}\left(\frac{\sin(\eta_n)}{1-\cos(\eta_n)}\right)^2,$$

where $\omega(\eta) = 2\sqrt{\frac{k}{m}}|\sin(\eta/2)|$ is the dispersion. The probability of no excitations is

$$\exp\left(-\sum_{n=1}^{N-1}\lambda_n\right) = \exp\left(-\sum_{n=1}^{N-1}\frac{4m\omega(\eta)\xi^2}{N}\left(\frac{\sin(\eta_n)}{1-\cos(\eta_n)}\right)^2\right).$$

Large N limit. Key point is that for small η we have a summand that goes like $1/|\eta_n|$ that will result in a logarithmic divergence. Specifically,

$$\frac{4m\omega(\eta)\xi^2}{N}\left(\frac{\sin(\eta_n)}{1-\cos(\eta_n)}\right)^2 \to \frac{16\sqrt{km}\xi^2}{N}\frac{1}{|\eta_n|} = \frac{8\sqrt{km}\xi^2}{\pi}\frac{1}{|n|},$$

so

$$\exp\left(-\sum_{n=1}^{N-1}\lambda_n\right) \sim \exp\left(-\frac{16\sqrt{km}\xi^2}{\pi}\log N\right).$$

For large but finite N the probability will vanish like a power law $N^{-\alpha}$ with

$$\alpha = \frac{16}{\pi}\left(\frac{\xi}{\ell_{\text{osc}}}\right)^2$$

in terms of the oscillator length $\ell_{\text{osc}} = (km)^{-1/4}$.

2 A system of one dimensional spinless fermions with periodic boundary conditions in a system of length L is described by the Hamiltonian

$$H = \sum_k \epsilon_k a_k^\dagger a_k + \overbrace{\frac{1}{2L} \sum_{k_1+k_2=k_3+k_4} V(k_1 - k_4) a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}}^{\equiv H_{\text{int}}},$$

where $\epsilon_k = k^2/2m$ and wavevectors are integer $\times \frac{2\pi}{L}$. The function $V(k)$ gives the Fourier components of the interaction between fermions. With $V = 0$ the eigenstates of H are product states $|\mathbf{N}\rangle$ described by vector of occupation numbers $\mathbf{N}$ with components $N_k = 0, 1$. The energy of state $|\mathbf{N}\rangle$ is $E_\mathbf{N} = \sum_k \epsilon_k N_k$.

Perturbation theory with perturbation H_{int} can be used to find how eigenstates change when $V \neq 0$. The first order correction is

$$|\mathbf{N}^{(1)}\rangle = \sum_{\mathbf{N}' \neq \mathbf{N}} \frac{\langle \mathbf{N}'|H_{\text{int}}|\mathbf{N}\rangle}{E_\mathbf{N} - E_{\mathbf{N}'}} |\mathbf{N}'\rangle.$$

(a) Consider the state $|\mathbf{N}\rangle = a_k^\dagger |\text{FS}\rangle$, where $|\text{FS}\rangle$ describes a filled Fermi sea ground state with Fermi wavevector k_F, and $k > k_\text{F}$. Show that two families of states $|\mathbf{N}'\rangle$ contribute to the first order correction $|\mathbf{N}^{(1)}\rangle$, the first being

$$|\mathbf{N}'\rangle = a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} |\text{FS}\rangle, \qquad |k_{1,2}| > k_\text{F}, |k_3| < k_\text{F}, \qquad k_1 + k_2 - k_3 = k. \quad \text{(F1)}$$

Give the general form of states in the second family. [8]

(b) We are interested in calculating the difference between $|\mathbf{N}^{(1)}\rangle$ and $a_k^\dagger |\text{FS}^{(1)}\rangle$, where $|\text{FS}^{(1)}\rangle$ is the correction to the Fermi sea ground state. Why can we ignore the second family of contributions? [6]

(c) Evaluate $\langle \mathbf{N}'|H_{\text{int}}|\mathbf{N}\rangle$ for the family (F1). [8]

(d) Analyze the behaviour of

$$\sum_{\mathbf{N}' \neq \mathbf{N}} \left| \frac{\langle \mathbf{N}'|H_{\text{int}}|\mathbf{N}\rangle}{E_\mathbf{N} - E_{\mathbf{N}'}} \right|^2$$

for the family (F1) as $L \to \infty$, paying attention to the regions where the denominator $E_\mathbf{N} - E_{\mathbf{N}'} = \epsilon_k + \epsilon_{k_3} - \epsilon_{k_1} - \epsilon_{k_2}$ is small and the numerator may be treated as constant.

[*Hint: consider separately the cases* $k_3 > 0$ *and* $k_3 < 0$] [8]

Solution 2

(a) The family (F1) corresponds to taking either $k_3 = k$ or $k_4 = k$ in H_{int}. The second family is

$$|\mathbf{N}'\rangle = a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} a_k |\text{FS}\rangle, \qquad |k_{1,2}| > k_\text{F}, |k_{3,4}| < k_\text{F}, \qquad k_1 + k_2 = k_3 + k_4 \quad (6)$$

(b) The correction to $|\text{FS}\rangle$ involves states $|\mathbf{N}''\rangle$

$$|\mathbf{N}''\rangle = a^\dagger_{k_1} a^\dagger_{k_2} a_{k_3} a_{k_4} |\text{FS}\rangle, \qquad |k_{1,2}| > k_\text{F}, |k_{3,4}| < k_\text{F}, \qquad k_1 + k_2 = k_3 + k_4 \quad (7)$$

and coefficients

$$\frac{\langle \mathbf{N}''|H_{\text{int}}|\text{FS}\rangle}{E_\text{FS} - E_{\mathbf{N}''}}$$

But $|\mathbf{N}'\rangle = a^\dagger_k |\mathbf{N}''\rangle$, so the correction to $a^\dagger_k|\text{FS}\rangle$ involves the same states. To confirm that the coefficients are the same, verify that

1. The numerator is the same, including the sign (accounting properly for anticommutation).
2. The denominator is the same, because the particle at k stays put in $|\mathbf{N}'\rangle$.

(c)

$$\langle \mathbf{N}'|H_{\text{int}}|\mathbf{N}\rangle = \frac{1}{2L} \sum_{k'_1+k'_2=k'_3+k'_4} V(k'_1 - k'_4) \langle \text{FS}| a^\dagger_{k_3} a_{k_2} a_{k_1} a^\dagger_{k'_1} a^\dagger_{k'_2} a_{k'_3} a_{k'_4} a^\dagger_k |\text{FS}\rangle \quad (8)$$

$$= L^{-1}(V(k_1 - k) - V(k - k_2)) \quad (9)$$

Evaluating the matrix element gives contributions from $k'_4 = k$ and $k'_3 = k$. The factor of two arises from the two assignments $k_1 = k'_1$, $k_2 = k'_2$ and $k_1 = k'_2$, $k_2 = k'_1$ (both leading to $k_3 = k'_3$).

(d) (i) $k_3 > 0$. The denominator vanishes when the created p-h pair (k_1, k_3) (say) has zero momentum, e.g. $k_1 - k_3 = k - k_2 = q \to 0$. Using the hint

$$\left(\epsilon_k + \epsilon_{k_3} - \epsilon_{k_1} - \epsilon_{k_2}\right)^2 \sim ((v_\text{F} - \epsilon'(k))q)^2$$

The numerator is then approximately

$$[V(k_\text{F} - k) - V(k - k_\text{F})]^2,$$

(ii) $k_3 < 0$. In this case the energy of a particle-hole pair near $-k_\text{F}$ can be arbitrarily small. The numerator is approximately

$$[V(-k_F - k) - V(k - k_\text{F})]^2,$$

while the denominator is the same as before

$$\left(\epsilon_k + \epsilon_{k_3} - \epsilon_{k_1} - \epsilon_{k_2}\right)^2 \sim ((v_\text{F} + \epsilon'(k))q)^2$$

where $q = k_2 - k = k_3 - k_1$.

Within this approximation there are $qL/2\pi$ identical terms corresponding to the placement of k_3 inside the Fermi surface and k_1 outside. Accounting for a factor of two that comes from switching k_1 and k_2 gives a (partial) sum for soft ph pairs in regions (i) and (ii)

$$\text{Region 1} = \frac{[V(k_\text{F} - k) - V(k - k_\text{F})]^2}{(v_\text{F} - \epsilon'(k))^2 \pi L} \sum_{k_\text{F} \gg q > 0} \frac{1}{q},$$

$$\text{Region 2} = \frac{[V(-k_F - k) - V(k - k_\text{F})]^2}{(v_\text{F} - \epsilon'(k))^2 \pi L} \sum_{k_\text{F} \gg q > 0} \frac{1}{q}$$

which are both logarithmically divergent. Bonus points: no UV problem with the integrals, even without the matrix element vanishing.

There are many places to go wrong with numerical factors, so I'll be looking for (i) correct dimensions (overall dimensionless), noting $[V(k)] = \text{Energy} \times \text{Length}$ (ii) appearance of $[V(2k_F) - V(0)]^2$, (iii) L dependence and (iv) log divergence.

3 Two systems of identical bosons are described by the Hamiltonian

$$H = \overbrace{\sum_k \left[\epsilon_{a,k} a_k^\dagger a_k + \epsilon_{b,k} b_k^\dagger b_k\right]}^{\equiv H_0} + \overbrace{v \sum_{k_1,k_2} \left[a_{k_1}^\dagger b_{k_2} + b_{k_2}^\dagger a_{k_1}\right]}^{\equiv H_t}.$$

The number difference operator $\mathcal{N}$ is defined as

$$\mathcal{N} = \frac{1}{2}\sum_k \left[a_k^\dagger a_k - b_k^\dagger b_k\right].$$

(a) Show that the Heisenberg equation of motion for $\mathcal{N}(t) = e^{iHt}\mathcal{N}e^{-iHt}$ is

$$\frac{d\mathcal{N}(t)}{dt} = vJ(t),$$

where $a_k(t) = e^{iHt}a_k e^{-iHt}$, $b_k(t) = e^{iHt}b_k e^{-iHt}$ and

$$J(t) \equiv -i \sum_{k_1,k_2} \left[a_{k_1}^\dagger(t) b_{k_2}(t) - b_{k_2}^\dagger(t) a_{k_1}(t)\right].$$

[6]

(b) When v is small we can approximate $a_k(t) \approx e^{iH_0t}a_k e^{-iH_0t} = e^{-i\epsilon_{a,k}t}a_k$, $b_k(t) \approx e^{iH_0t}b_k e^{-iH_0t} = e^{-i\epsilon_{b,k}t}b_k$. With this approximation evaluate $\langle N_a, N_b|J(t)|N_a, N_b\rangle$ and $\langle N_a, N_b|J(t)J(0)|N_a, N_b\rangle$ for the state

$$|N_a, N_b\rangle = \frac{1}{\sqrt{N_a!N_b!}}\left(a_0^\dagger\right)^{N_a}\left(b_0^\dagger\right)^{N_b}|\mathrm{VAC}\rangle.$$

[7]

(c) Using the same approximation show that in a product state with occupancies $N_{a,k}$ and $N_{b,k}$

$$\langle \mathbf{N}_a, \mathbf{N}_b|\frac{dJ(t)}{dt}|\mathbf{N}_a, \mathbf{N}_b\rangle = 2v\sum_{q_1,q_2}\left(N_{b,q_2} - N_{a,q_1}\right)\cos\left[(\epsilon_{a,q_1} - \epsilon_{b,q_2})t\right]. \qquad (\star)$$

[10]

(d) Now suppose $v(t) = ve^{\delta t}$ for $-\infty < t \le 0$ in H_t. Modify ($\star$) accordingly and integrate to find $\langle J(0)\rangle$. Assume that $\delta \to 0$ but $\delta \gg \epsilon_{a/b,k+1} - \epsilon_{a/b,k}$, the level spacings of the a and b systems. Interpret the result for $\langle d\mathcal{N}(0)/dt\rangle$.

[*Hint: the formula* $\lim_{\delta\to 0}\frac{\delta}{\epsilon^2+\delta^2} = \pi\delta(\epsilon)$ *may be useful*]

[7]

Solution 3

(a) The equation of motion implies

$$\frac{d\mathcal{N}(t)}{dt} = i[H, \mathcal{N}(t)] = ie^{iHt}[H, \mathcal{N}]e^{-iHt} = ie^{iHt}[H_\mathrm{t}, \mathcal{N}]e^{-iHt} \tag{10}$$

(because H_0 conserves the number difference). Computing the commutator gives

$$[H_\mathrm{t}, \mathcal{N}] = v \sum_{k_1,k_2} \left[b^\dagger_{k_1} a_{k_2} - a^\dagger_{k_2} b_{k_1} \right], \tag{11}$$

which leads to the stated result.

(b) $\langle N_a, N_b | J(t) | N_a, N_b \rangle = 0$. It's fine to state it with the reason that J transfers a particle so takes us to an orthogonal state.

$$\langle J(t)J(0)\rangle = -\sum_{k_1,k_2} \langle \left[a^\dagger_{k_1}(t) b_{k_2}(t) - b^\dagger_{k_2}(t) a_{k_1}(t) \right] \left[a^\dagger_{k_1}(0) b_{k_2}(0) - b^\dagger_{k_2}(0) a_{k_1}(0) \right] \rangle \tag{12}$$

$$= \sum_{k_1,k_2} \langle a^\dagger_{k_1}(t) b_{k_2}(t) b^\dagger_{k_2}(0) a_{k_1}(0) + b^\dagger_{k_2}(t) a_{k_1}(t) a^\dagger_{k_1}(0) b_{k_2}(0) \rangle \tag{13}$$

$$= \sum_k \left[e^{i(\epsilon_{a,0}-\epsilon_{b,k})t} N_a (N_b \delta_{k,0} + 1) + e^{i(\epsilon_{b,0}-\epsilon_{a,k})t} N_b (N_a \delta_{k,0} + 1) \right] \tag{14}$$

(c) The equation of motion is

$$\frac{dJ(t)}{dt} = i[H, J(t)] = i[H_0, J(t)] + i[H_\mathrm{t}, J(t)]. \tag{15}$$

Taking the expectation and using $\langle [H_0, J(t)] \rangle = 0$ (which holds in an eigenstate of the unperturbed problem and *must be justified*) we have

$$\langle \frac{dJ(t)}{dt} \rangle = i \langle [H_\mathrm{t}, J(t)] \rangle.$$

The commutator is

$$[H_\mathrm{t}, J(t)] = -iv^2 e^{\delta t} \sum_{k_1,k_2,q_1,q_2} \left(\left[b^\dagger_{k_2} a_{k_1}, a^\dagger_{q_1}(t) b_{q_2}(t) \right] - \left[a^\dagger_{k_1} b_{k_2}, b^\dagger_{q_1}(t) a_{q_2}(t) \right] \right) \tag{16}$$

$$= -iv^2 e^{\delta t} \sum_{k_1,k_2,q_1,q_2} \left[\left(b^\dagger_{k_2} b_{q_2} \delta_{k_1,q_1} - a^\dagger_{q_1} a_{k_1} \delta_{k_2,q_2} \right) e^{i(\epsilon_{a,q_1}-\epsilon_{b,q_2})t} \right. \tag{17}$$

$$\left. - \left(a^\dagger_{k_1} a_{q_2} \delta_{k_2,q_1} - b^\dagger_{q_1} b_{k_2} \delta_{k_1,q_2} \right) e^{i(\epsilon_{b,q_1}-\epsilon_{a,q_2})t} \right] \tag{18}$$

A

Taking the expectation values using

$$\langle a_k^\dagger a_q \rangle = N_{a,k}\delta_{k,q}$$

and similarly for $\langle b_k^\dagger b_q \rangle$ gives

$$i\langle [H_t, J(t)]\rangle = v^2 e^{\delta t} \sum_{q_1,q_2} \Big[\left(N_{b,q_2} - N_{a,q_1}\right) e^{i(\epsilon_{a,q_1} - \epsilon_{b,q_2})t} - \left(N_{a,q_2} - N_{b,q_1}\right) e^{i(\epsilon_{b,q_1} - \epsilon_{a,q_2})t} \Big]. \quad (19)$$

(d) Doing the integral

$$\int_{-\infty}^{0} e^{\delta t + i\epsilon t} dt = \frac{1}{\delta + i\epsilon}$$

gives eventually

$$\int_{-\infty}^{0} i\langle [H_t, J(t)]\rangle dt = v^2 \sum_{k,q} N_{b,k} \frac{2\delta}{\delta^2 + (\epsilon_{a,q} - \epsilon_{b,k})^2} - N_{a,k} \frac{2\delta}{\delta^2 + (\epsilon_{a,k} - \epsilon_{b,q})^2} \quad (20)$$

$$= 2\pi v^2 \sum_k \frac{N_{b,k}}{\varDelta_a} - \frac{N_{a,k}}{\varDelta_b}, \quad (21)$$

or

$$\langle J(0)\rangle = 2\pi v^2 \sum_k \frac{N_{b,k}}{\varDelta_a} - \frac{N_{a,k}}{\varDelta_b}$$

Interpretation: Golden Rule rate $2\pi\delta(\epsilon)v^2$ for each level weighted by occupation $N_{a/b,k}$ gives average current in each direction, with difference giving net current.

END OF PAPER