UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS: Part III Physics — NST3PHY
MASTER OF ADVANCED STUDY IN PHYSICS — MAPY

20 January 2022, 10.00-12.00

MAJOR TOPICS
Paper 1/TQM (Theories of Quantum Matter)

Answer **two** *questions only.* $\hbar = 1$ ***throughout this paper***.

The approximate number of marks allocated to each part of a question is indicated in the right-hand margin where appropriate.

The paper contains 4 sides including this one and is accompanied by a book giving values of constants and containing mathematical formulae which you may quote without proof.
You should use a **separate Answer Book** *for each question.*

STATIONERY REQUIREMENTS
linear graph paper
Rough workpad

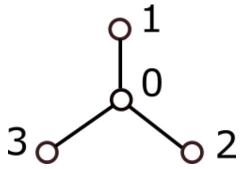
SPECIAL REQUIREMENTS
Mathematical Formulae Handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

V8.4

A

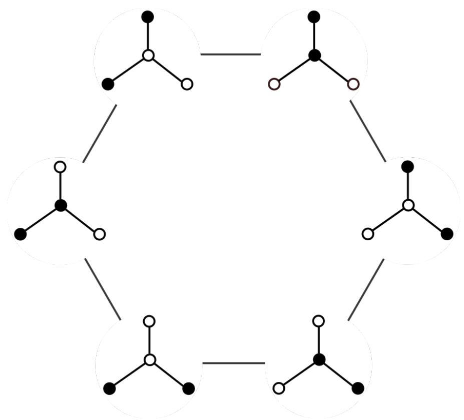
1 A system of four sites is connected as follows



The Hamiltonian is

$$H = -t \sum_{<jk>} \left[a_j^\dagger a_k + a_k^\dagger a_j \right],$$

where $\sum_{<ij>}$ denotes a sum over bonds. In this question you will consider particles that are either (i) hardcore bosons satisfying $(a_j^\dagger)^2 = 0$ or (ii) fermions (without spin). In both cases the nonzero matrix elements of the Hamiltonian for two particles correspond to edges of the graph below (filled circles denote particles):

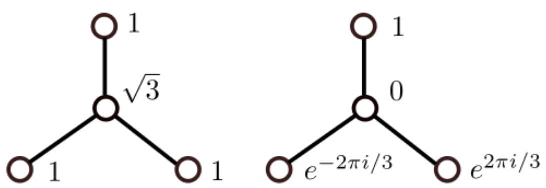


Using this graph:

(a) Find the energy spectrum (including degeneracies) for two hardcore bosons. [8]

(b) Find the energy spectrum (including degeneracies) for two fermions.
[*You will need to assign signs carefully to the basis states* $a_j^\dagger a_k^\dagger |VAC\rangle$] [12]

Two of the single particle energy eigenstates are



(c) Find the remaining two single particle energy eigenstates and the four single particle energy eigenvalues. [6]

(d) By filling single particle states according to Fermi statistics, find the spectrum and compare your answer to part (b). [4]

Solution 1

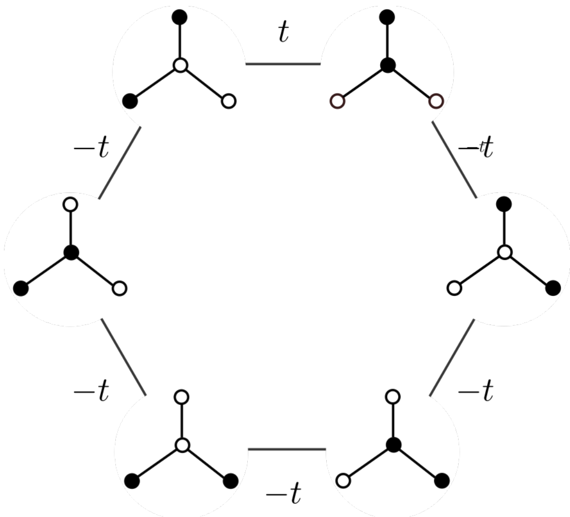
(a) The matrix elements are all $-t$. The Hamiltonian then takes the form of a circulant matrix and the eigenstates are plane waves with wavevectors $\eta = 0$, $\pm 2\pi/6$, $\pm 4\pi/6$, and π with corresponding energies $E(\eta) = -2t\cos(\eta)$ i.e. $E = -2t, -t$ (degeneracy 2), $+t$ (degeneracy 2), $+2t$.

The above reasoning is adequate, though candidates may take longer to establish the above facts.

(b) The key here is to get the minus signs right! One possible basis is

$$
\begin{aligned}
|1\rangle &= a_0^\dagger a_1^\dagger |\mathrm{VAC}\rangle \\
|2\rangle &= a_2^\dagger a_1^\dagger |\mathrm{VAC}\rangle \\
|3\rangle &= a_2^\dagger a_0^\dagger |\mathrm{VAC}\rangle \\
|4\rangle &= a_2^\dagger a_3^\dagger |\mathrm{VAC}\rangle \\
|5\rangle &= a_0^\dagger a_3^\dagger |\mathrm{VAC}\rangle \\
|6\rangle &= a_1^\dagger a_3^\dagger |\mathrm{VAC}\rangle
\end{aligned} \tag{1}
$$

which leads to the signs illustrated below

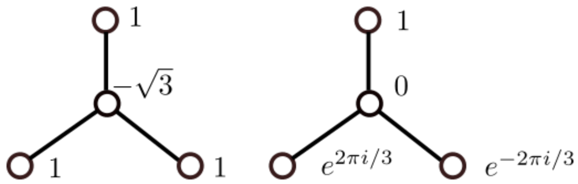


Other choices of basis will generate pairs of sign changes on the edges on either side of vertex. Pairs can be eliminated by changing the signs on a vertex until we have the above configuration. The eigenstates then again correspond to plane waves but with *anti*-periodic boundary conditions i.e. $\eta = \pm\pi/6, \pm 3\pi/6, \pm 5\pi/6$ with $E(\eta) = -2t\cos(\eta)$ given by $-t\sqrt{3}$ (degeneracy 2), 0 (degeneracy 2), and $+t\sqrt{3}$ (degeneracy 2).

(c) The single particle Hamiltonian is

$$H_{\text{sp}} = -t \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The two given eigenstates have energy $-t\sqrt{3}$ and 0 respectively. The other two are



with energy $t\sqrt{3}$ and 0.

(d) By filling the single particles states with energies $-t\sqrt{3}, 0, 0, +t\sqrt{3}$ with two fermions, we get the same energies as in (b)

$$\begin{aligned} -t\sqrt{3} + 0 &= -t\sqrt{3} \\ -t\sqrt{3} + 0 &= -t\sqrt{3} \\ 0 + 0 &= 0 \\ -t\sqrt{3} + t\sqrt{3} &= 0 \\ t\sqrt{3} + 0 &= t\sqrt{3} \\ t\sqrt{3} + 0 &= t\sqrt{3} \end{aligned} \quad (2)$$

What's been seen before?

- Candidates have met many examples of a circulant matrix being diagonalized by plane waves, but the setting of parts (a) and (b) is novel.
- Filling the single particle states for a Fermi system (parts (c) and (d)) is core knowledge.

2 This question concerns the N-site spin-s anisotropic spin chain Hamiltonian

$$H = -\sum_j \left[J_{\parallel} s_j^z s_{j+1}^z + J_{\perp}\left(s_j^x s_{j+1}^x + s_j^y s_{j+1}^y\right)\right],$$

where $J_{\parallel} > J_{\perp} > 0$, and $\mathbf{s}_j = (s_j^x, s_j^y, s_j^z)$ obey the usual angular momentum commutation relations and we assume periodic boundary conditions: $\mathbf{s}_{N+1} = \mathbf{s}_1$.

(a) The Holstein–Primakoff representation is

$$s_j^+ = \sqrt{2s}\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} a_j$$

$$s_j^- = \sqrt{2s}a_j^\dagger\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2}$$

$$s_j^z = \left(s - a_j^\dagger a_j\right).$$

where $s_j^\pm = s_j^x \pm i s_j^y$ and $a_j^\dagger$, a_j satisfy the standard bosonic commutation relations $\left[a_j, a_k^\dagger\right] = \delta_{jk}$. By approximating H by a Hamiltonian quadratic in the $a_j^\dagger$, a_j, show that the operators

$$a_\eta = \frac{1}{\sqrt{N}}\sum_j a_j \exp\left(-i\eta j\right),$$

diagonalize this Hamiltonian, where η takes the values $\eta = (2\pi/N) \times$ integer. Show that the dispersion relation $\omega(\eta)$ of the (magnon) excitations is

$$\omega(\eta) = 2s\left(J_{\parallel} - J_{\perp}\cos(\eta)\right)$$ [12]

According to the quadratic Hamiltonian, the ground state of a system with total $S^z = sN - n$, corresponds to placing n magnons at $\eta = 0$. In the rest of this question we explore the consequences of including higher order terms in the Hamiltonian. The quartic terms have the form

$$H_4 = \sum_{\substack{\eta_1,\eta_2,\eta_3,\eta_4 \\ \eta_1+\eta_2=\eta_3+\eta_4}} U(\eta_1,\eta_2,\eta_3,\eta_4) a_{\eta_1}^\dagger a_{\eta_2}^\dagger a_{\eta_3} a_{\eta_4}$$

$$U(\eta_1,\eta_2,\eta_3,\eta_4) \equiv \frac{1}{N}\left(-J_{\parallel}\cos(\eta_1 - \eta_3) + \frac{J_{\perp}}{4}\left[\cos(\eta_1) + \cos(\eta_2) + \cos(\eta_3) + \cos(\eta_4)\right]\right).$$

(b) Show that in the presence of n magnons at $\eta = 0$ the quartic Hamiltonian H_4 gives rise (in the Bogoliubov approximation) to the new quadratic terms

$$H_4 \to n\sum_{\eta\neq 0}\left[U_1 a_\eta^\dagger a_\eta + U_2\left(a_\eta^\dagger a_{-\eta}^\dagger + a_\eta a_{-\eta}\right)\right].$$

Give expressions for $U_{1,2}$ in terms of the function $U(\eta_1,\eta_2,\eta_3,\eta_4)$. [10]

The contribution of the quadratic Hamiltonian

$$H_2 \to \sum_{\eta \neq 0} \left[\omega(\eta) - \omega(0)\right] a_\eta^\dagger a_\eta \qquad (\star)$$

may be combined with the quadratic terms from part (b) to give a Bogoliubov Hamiltonian.

(c) Explain why $\omega(0) \sum_{\eta \neq 0} a_\eta^\dagger a_\eta$ has been subtracted in ($\star$). [2]

The Bogoliubov Hamiltonian

$$H_{\mathrm{B}} = \sum_{\eta \neq 0} \left(A(\eta) a_\eta^\dagger a_\eta + \frac{B(\eta)}{2} \left[a_\eta^\dagger a_{-\eta}^\dagger + a_\eta a_{-\eta} \right] \right)$$

has excitations with dispersion relation $\Omega(\eta) = \sqrt{A(\eta)^2 - B(\eta)^2}$.

(d) At small η $A(\eta)$ and $B(\eta)$ are found to have the form

$$\begin{aligned} A(\eta) &= \frac{2n}{N}\left(J_\perp - J_\parallel\right) + \eta^2 \left(\left[s - \frac{n}{N} \right] J_\perp + \frac{n}{N} J_\parallel \right) + \cdots \\ B(\eta) &= \frac{2n}{N}\left(J_\perp - J_\parallel\right) + \eta^2 \frac{n}{N} \left(J_\parallel - \frac{J_\perp}{2} \right) + \cdots \end{aligned}$$

What can you deduce about the stability of the ‘magnon condensate’? [6]

Solution 2

(a) The lowest order approximation (valid at large s) coming from the HP representation is

$$s_j^+ = \sqrt{2s}a \qquad s_j^- = \sqrt{2s}a^\dagger \tag{3}$$
$$s_j^z = \left(s - a^\dagger a\right). \tag{4}$$

Write the Hamiltonian in the form

$$H = -\sum_j \left[J_\parallel s_j^z s_{j+1}^z + \frac{J_\perp}{2}\left(s_j^+ s_{j+1}^- + s_{j+1}^+ s_j^-\right)\right].$$

Substituting in the HP representation gives

$$H = -\sum_j \left[J_\parallel (s - a_j^\dagger a_j)(s - a_{j+1}^\dagger a_{j+1})\right.$$
$$\left. + J_\perp \left[s\left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2} a_j a_{j+1}^\dagger \left(1 - \frac{a_{j+1}^\dagger a_{j+1}}{2s}\right)^{1/2} + s\left(1 - \frac{a_{j+1}^\dagger a_{j+1}}{2s}\right)^{1/2} a_{j+1} a_j^\dagger \left(1 - \frac{a_j^\dagger a_j}{2s}\right)^{1/2}\right]\right] \tag{5}$$

The lowest order approximation corresponds to taking the square roots to be 1. This yields the zeroth order term $-NJ_\parallel s^2$ (not required) and the quadratic Hamiltonian

$$H_2 = 2s\sum_j \left[J_\parallel a_j^\dagger a_j - \frac{J_\perp}{2}\left(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j\right)\right].$$

Substitute in the given mode expansion and sum over j, giving

$$H_2 = \sum_\eta \omega(\eta) a_\eta^\dagger a_\eta$$

where $\omega(\eta)$ is the quoted dispersion.

(b) Following the Bogoliubov procedure of replacing the zero wavevector operators $a_0^\dagger = a_0 = \sqrt{n}$ in the quadratic term gives

$$H_4 \to U(0,0,0,0) a_0^\dagger a_0^\dagger a_0 a_0 + n \sum_{\eta\neq 0} \left[U(\eta,-\eta,0,0) a_\eta^\dagger a_{-\eta}^\dagger + U(0,0,\eta,-\eta) a_\eta a_{-\eta} \right. \tag{6}$$

$$\left. + \left(U(\eta,0,0,\eta) + U(0,\eta,\eta,0) + U(\eta,0,\eta,0) + U(0,\eta,0,\eta)\right) a_\eta^\dagger a_\eta \right] \tag{7}$$

$$= U(0,0,0,0) n^2 + n \sum_{\eta\neq 0} \left[\overbrace{U(\eta,-\eta,0,0)}^{\equiv U_2} a_\eta^\dagger a_{-\eta}^\dagger + U(0,0,\eta,-\eta) a_\eta a_{-\eta} \right. \tag{8}$$

$$\left. + \overbrace{\left(U(\eta,0,0,\eta) + U(0,\eta,\eta,0) + U(\eta,0,\eta,0) + U(0,\eta,0,\eta) - 2U(0,0,0,0)\right)}^{\equiv U_1} a_\eta^\dagger a_\eta \right] \tag{9}$$

$$= \frac{n}{N} \sum_{\eta\neq 0} \left[\left(-J_\parallel \cos\eta + \frac{J_\perp}{2}(\cos(\eta)+1) \right) \left(a_\eta^\dagger a_{-\eta}^\dagger + a_\eta a_{-\eta} \right) \right. \tag{10}$$

$$\left. + 2\cos\eta \left(-J_\parallel + J_\perp\right) a_\eta^\dagger a_\eta \right] \tag{11}$$

(Note that only the expressions for $U_{1,2}$ in terms of $U(\eta_1, \eta_2, \eta_3, \eta_4)$ are required at this stage) We have handled the first term by writing $n = a_0^\dagger a_0 + \sum_{\eta\neq 0} a_\eta^\dagger a_\eta$ and dropping terms that are not quadratic in $a_0^\dagger$, a_0.

(c) The shift is obtained by again using $n = a_0^\dagger a_0 + \sum_{\eta\neq 0} a_\eta^\dagger a_\eta$ to write

$$H_2 = \sum_\eta \omega(\eta) a_\eta^\dagger a_\eta = n\omega(0) + \sum_{\eta\neq 0} \left[\omega(\eta) - \omega(0)\right] a_\eta^\dagger a_\eta.$$

The first term is a constant.

(d) (For completeness we give more details here than are needed) The Bogoliubov Hamiltonian is

$$H_2 = \sum_{\eta\neq 0} \left(\left[\omega(\eta) - \omega(0) + U_1\right] a_\eta^\dagger a_\eta + U_2 \left(a_\eta^\dagger a_{-\eta}^\dagger + a_\eta a_{-\eta} \right) \right). \tag{12}$$

Using the given dispersion

$$\Omega^2(\eta) = A_\eta^2 - B_\eta^2 \tag{13}$$

$$= \left[\omega(\eta) - \omega(0) + U_1\right]^2 - 4U_2^2 \tag{14}$$

$$= \left[2sJ_\perp(1-\cos\eta) + \frac{2n}{N}\cos\eta \left(-J_\parallel + J_\perp\right) \right]^2 \tag{15}$$

$$- \left[\frac{2n}{N} \left(-J_\parallel \cos\eta + \frac{J_\perp}{2}(\cos(\eta)+1) \right) \right]^2. \tag{16}$$

Expanding A_η and B_η in η yields the given forms.

Considering the region of small η shows that $\Omega(0) = 0$ (expected on conservation grounds). At quadratic order we have

$$\Omega^2(\eta) \sim \eta^2 \frac{4n}{N} J_\perp \left(J_\perp - J_\|\right) \left(s - \frac{n}{2N}\right).$$

Since $J_\perp < J_\|$ by assumption this shows the 'magnon condensate' is unstable. Essentially this is because the magnons attract each other.

What's been seen before?

- Magnon dispersion in the isotropic case is discussed in lectures; the anisotropic case in the first Problem Set.
- The manipulations in deriving the Bogoliubov Hamiltonian in parts (b) and (c) have been seen in the context of the Bose gas. The spin chain setting is new.

3 A model for a magnetic impurity in a metal is given by the Hamiltonian

$$H = H_{\text{metal}} + H_{\text{imp}} + H_{\text{t}}$$
$$H_{\text{metal}} = \sum_{\mathbf{k},\sigma} \xi(\mathbf{k}) c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}$$
$$H_{\text{imp}} = \epsilon_0 n + \frac{U}{2} n(n-1) \qquad n \equiv \sum_{\sigma} a^{\dagger}_{\sigma} a_{\sigma}$$
$$H_{\text{t}} = \sum_{\mathbf{k},\sigma} t_{\mathbf{k}} \left(a^{\dagger}_{\sigma} c_{\mathbf{k},\sigma} + c^{\dagger}_{\mathbf{k},\sigma} a_{\sigma} \right)$$

$c_{\mathbf{k},\sigma}$, $c^{\dagger}_{\mathbf{k},\sigma}$ describe an electron with momentum $\mathbf{k}$ and spin σ. a_{σ}, $a^{\dagger}_{\sigma}$ describe an electron on the impurity site with spin σ. $\xi(\mathbf{k}) = \mathbf{k}^2/2m - E_{\text{F}}$ is the electron energy relative to the Fermi energy E_{F}. ϵ_0 is the energy of the impurity state. $U > 0$ describes repulsion between particles in the impurity state. $t_{\mathbf{k}}$ is a coupling between the metal and impurity states.

(a) For $t_{\mathbf{k}} = 0$ (no coupling between the impurity and the metal), find the range of ϵ_0 for which the ground state has a single electron in the impurity state ($n = 1$). [2]

In first order perturbation theory in $t_{\mathbf{k}}$ the correction to the unperturbed energy eigenstate $|a\rangle$ is given by

$$\sum_{b \neq a} \frac{\langle b|H_{\text{t}}|a\rangle}{E^{(0)}_a - E^{(0)}_b} |b\rangle \, .$$

(b) Show that when $n = 1$ the correction to the ground state $|n = 1, \sigma\rangle$ has the form

$$\left[\sum_{|\mathbf{k}|>k_{\text{F}}} \alpha_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\sigma} a_{\sigma} + \sum_{|\mathbf{k}|<k_{\text{F}}} \beta_{\mathbf{k}} a^{\dagger}_{-\sigma} c_{\mathbf{k},-\sigma} \right] |n = 1, \sigma\rangle \, .$$

Give expressions for $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$ and identify k_{F}. [6]

The second order correction to the energy of the eigenstate $|a\rangle$ is given by

$$\sum_{b \neq a} \frac{|\langle b|H_{\text{t}}|a\rangle|^2}{E^{(0)}_a - E^{(0)}_b} .$$

(c) Evaluate the second order correction to the energy of the $n = 1$ ground state, assuming that $t_{\mathbf{k}}$ has the form

$$t_{\mathbf{k}} = \begin{cases} t & |\xi(\mathbf{k})| \leq \Lambda \\ 0 & |\xi(\mathbf{k})| > \Lambda \end{cases}$$

for some $\epsilon_0, U \ll \Lambda \ll E_{\text{F}}$. You may treat the density of states as constant for $|\xi(\mathbf{k})| \leq \Lambda$. [8]

(d) Using the variational wavefunction

$$|z, \alpha_{\mathbf{k}}, \beta_{\mathbf{k}}\rangle = \left[z + \sum_{|\mathbf{k}|>k_{\mathrm{F}}} \alpha_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\sigma} a_{\sigma} + \sum_{|\mathbf{k}|<k_{\mathrm{F}}} \beta_{\mathbf{k}} a^{\dagger}_{-\sigma} c_{\mathbf{k},-\sigma} \right] |n = 1, \sigma\rangle$$

show that the minimization of the expectation of the $n = 1$ ground state energy can be reduced to the solution of the equation

$$\mathit{\Delta} = \sum_{|\mathbf{k}|>k_{\mathrm{F}}} \frac{t_{\mathbf{k}}^2}{\xi(\mathbf{k}) - \epsilon_0 + \mathit{\Delta}} + \sum_{|\mathbf{k}|<k_{\mathrm{F}}} \frac{t_{\mathbf{k}}^2}{-\xi(\mathbf{k}) + U + \epsilon_0 + \mathit{\Delta}},$$

where you should identify the quantity $\mathit{\Delta}$ in terms of z, $\alpha_{\mathbf{k}}$, and $\beta_{\mathbf{k}}$.

[*Normalization is best handled with a Lagrange multiplier!*] [10]

(e) Discuss the behaviour of $\mathit{\Delta}$ as ϵ_0 approaches the ends of the range found in (a). [4]

Solution 3

(a) The energies of the impurity 0, ϵ_0 and $2\epsilon_0 + U$ for zero, one, and two electrons respectively (any changes in the ground state of the metal are negligible in the thermodynamic limit). Thus $n = 1$ when $-U < \epsilon_0 < 0$.

(b) We need to compute the nonzero matrix elements. H_t either removes an electron from the impurity and adds an electron above the Fermi sea (i.e. with $|\mathbf{k}| > k_F$, where k_F is the Fermi wavevector), giving a state

$$c^\dagger_{\mathbf{k},\sigma} a_\sigma |n = 1, \sigma\rangle, \qquad |\mathbf{k}| > k_F,$$

or adds and electron to the impurity and removes an electron below the Fermi sea (with $|\mathbf{k}| < k_F$ i.e. creates a hole).

$$a^\dagger_{-\sigma} c_{\mathbf{k},-\sigma} |n = 1, \sigma\rangle, \qquad |\mathbf{k}| < k_F.$$

In both cases the matrix elements are $t_\mathbf{k}$. Evaluating the energy denominators gives

$$\alpha_\mathbf{k} = \frac{t_\mathbf{k}}{\epsilon_0 - \xi(\mathbf{k})} \qquad \beta_\mathbf{k} = \frac{t_\mathbf{k}}{\xi(\mathbf{k}) - U - \epsilon_0}.$$

Note that both are negative.

(c) Having computed the matrix element we can write down the expression for the energy straight away

$$-\sum_{|\mathbf{k}|>k_F} \frac{t_\mathbf{k}^2}{\xi(\mathbf{k}) - \epsilon_0} + \sum_{|\mathbf{k}|<k_F} \frac{t_\mathbf{k}^2}{\xi(\mathbf{k}) - U - \epsilon_0}.$$

This is negative. If we now add the assumptions about $t_\mathbf{k}$ we get

$$-\nu t^2 \int_0^\Lambda \frac{1}{\xi - \epsilon_0} d\xi + \nu t^2 \int_{-\Lambda}^0 \frac{1}{\xi - U - \epsilon_0} d\xi = -\nu t^2 \left[\log\left(\frac{\Lambda}{-\epsilon_0}\right) + \log\left(\frac{\Lambda}{U + \epsilon_0}\right)\right]. \tag{17}$$

where ν is the Fermi surface density of states per unit volume (per spin).

(d) Introduce a Lagrange multiplier for the normalization

$$\mathcal{N}(z, \alpha_\mathbf{k}, \beta_\mathbf{k}) = |z|^2 + \sum_{|\mathbf{k}|>k_F} |\alpha_\mathbf{k}|^2 + \sum_{|\mathbf{k}|<k_F} |\beta_\mathbf{k}|^2,$$

and minimize the quantity $\langle z, \alpha_\mathbf{k}, \beta_\mathbf{k}|H|z, \alpha_\mathbf{k}, \beta_\mathbf{k}\rangle - \lambda\mathcal{N}(z, \alpha_\mathbf{k}, \beta_\mathbf{k})$. The expectation of the Hamiltonian is

$$\langle z, \alpha_\mathbf{k}, \beta_\mathbf{k}|H|z, \alpha_\mathbf{k}, \beta_\mathbf{k}\rangle = \sum_{|\mathbf{k}|>k_F} \xi(\mathbf{k})|\alpha_\mathbf{k}|^2 + t_\mathbf{k}\left(z\alpha^*_\mathbf{k} + \text{c.c.}\right) \tag{18}$$

$$+ \sum_{|\mathbf{k}|<k_F} \left(-\xi(\mathbf{k}) + U + 2\epsilon_0\right)|\beta_\mathbf{k}|^2 + t_\mathbf{k}\left(z\beta^*_\mathbf{k} + \text{c.c.}\right) \tag{19}$$

$$+ |z|^2\epsilon_0 \tag{20}$$

Differentiating $\langle z, \alpha_{\mathbf{k}}, \beta_{\mathbf{k}} | H | z, \alpha_{\mathbf{k}}, \beta_{\mathbf{k}} \rangle - \lambda \mathcal{N}(z, \alpha_{\mathbf{k}}, \beta_{\mathbf{k}})$ gives

$$\begin{aligned} [\xi(\mathbf{k}) - \lambda]\, \alpha_{\mathbf{k}} + t_{\mathbf{k}} z &= 0 \\ [-\xi(\mathbf{k}) - \lambda + U + 2\epsilon_0]\, \beta_{\mathbf{k}} + t_{\mathbf{k}} z &= 0 \\ z\, \overbrace{(\epsilon_0 - \lambda)}^{\equiv \varDelta} + \sum_{|\mathbf{k}|>k_{\mathrm{F}}} t_{\mathbf{k}} \alpha_{\mathbf{k}} + \sum_{|\mathbf{k}|<k_{\mathrm{F}}} t_{\mathbf{k}} \beta_{\mathbf{k}} &= 0. \end{aligned} \tag{21}$$

With the above definition of $\varDelta$ we have

$$\alpha_{\mathbf{k}} = \frac{z t_{\mathbf{k}}}{\epsilon_0 - \xi(\mathbf{k}) - \varDelta} \qquad \beta_{\mathbf{k}} = \frac{z t_{\mathbf{k}}}{\xi(\mathbf{k}) - U - \epsilon_0 - \varDelta},$$

and the final equation gives the quoted result. In terms of the variational parameters $\varDelta$ is

$$z\varDelta = -\sum_{|\mathbf{k}|>k_{\mathrm{F}}} t_{\mathbf{k}} \alpha_{\mathbf{k}} - \sum_{|\mathbf{k}|<k_{\mathrm{F}}} t_{\mathbf{k}} \beta_{\mathbf{k}}.$$

(e) As ϵ_0 approaches 0 or $-U$ the correction to the energy computed in (c) diverges logarithmically. This is not the case in the variational approach. As the ends of the range are approached a positive $\varDelta$ prevents a divergence, showing that $\varDelta > 0$ is always a solution no matter how small the coupling $t_{\mathbf{k}}$.

What's been seen before?

- Perturbation theory for the uniform Fermi gas is treated in the lectures. The application to an impurity problem is new.

END OF PAPER